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MINIMIZATION OF THE DENSE SUBSET

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Abstract. We introduced the concept of the ϵ_0-density and the ϵ_0-dense ace in [1]. This concept is related to the structure of employment. In addition to the double capacity theorem which was introduced in [1], we need the minimal dense subset. In this paper, we investigate a concept of the minimal ϵ_0- dense subset in the Euclidean m dimensional space.

1. Introduction

In this section, we introduce a concept of the locally finite ϵ_0-dense subset in the space R^m. And we study some properties of this concept which we need later. Throughout this paper, $\epsilon_0 \geq 0$ denotes any, but fixed, non-negative real number. We denote the open and closed balls with radius ϵ and center at α in the space R^m by $B(\alpha,\epsilon) = \{x \in R^m : \|x-\alpha\| < \epsilon\}$ and $\overline{B}(\alpha,\epsilon) = \{x \in R^m : \|x-\alpha\| \leq \epsilon\}$.

Definition 1.1. Let S be a subset of R^m. A point $a \in R^m$ is called an ϵ_0-accumulation point of the subset S if and only if $B(a,\epsilon)\cap(S-\{a\}) \neq \emptyset$ for all $\epsilon > \epsilon_0$. And a point $a \in S$ is called an ϵ_0-isolated point of S if and only if $B(a,\epsilon_1) \cap (S-\{a\}) = \emptyset$ for some positive number $\epsilon_1 > \epsilon_0$.

Definition 1.2. For a subset S of R^m, the set of all the ϵ_0-accumulation points of S is called the ϵ_0-derived set of S and denote it by $S'_{(\epsilon_0)}$.

Definition 1.3. Let E be any non-empty and open subset of R^m and $\epsilon_0 \geq 0$. And let a subset D of E be given. D is called an ϵ_0-dense subset of E in E if and only if $E \subseteq D'_{(\epsilon_0)} \cup D$. In this case, we say that D is ϵ_0-dense in E.

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Definition 1.4. Let E be an open non-empty subset of R^m. And let D be an ϵ_0-dense subset of E in E. An element $a \in D$ is called a point of the ϵ_0-dense ace of D in E if and only if $D - \{a\}$ is not ϵ_0-dense in E.

Definition 1.5. Let D be a subset of R^m. The set D is called a locally finite subset if and only if $D \cap B(x, \epsilon)$ is a finite subset of R^m for each positive real number $\epsilon > 0$ and all $x \in R^m$.

The following lemmas 1.6, 1.7, 1.8 and corollary 1.9 are proved in [1]; theorem 2.10 $\sim$ theorem 2.13, and we omit the referred proofs.

Lemma 1.6. *Let E be an open subset of R^m and D be a non-empty subset of E. Suppose that $E \subseteq \underset{b \in D}{\cup} \overline{B}(b, \epsilon_0)$. Then D is ϵ_0-dense in E.*

Lemma 1.7. *Let D be a non-empty subset of an open subset E of R^m and $\overline{D} = D'_{(0)} \cup D$. Then D is ϵ_0-dense in E if and only if $E \subseteq \underset{b \in \overline{D}}{\cup} \overline{B}(b, \epsilon_0)$.*

Lemma 1.8. *Let D be a subset of an open subset E of R^m and $\epsilon_0 \geq 0$ be any, but fixed, non-negative real number. Then D is ϵ_0-dense in E if and only if $E \subseteq \underset{b \in D}{\cup} \overline{B}(b, \epsilon)$ for each positive real number $\epsilon > \epsilon_0$.*

Corollary 1.9. *Let D be a subset of an open subset E of R^m and $\epsilon_0 \geq 0$ be any, but fixed, non-negative real number. Then D is not ϵ_0-dense in E if and only if we have $B(a_1, \epsilon_1) \cap D = \emptyset$ for some positive real number $\epsilon_1 > \epsilon_0$ and some vector $a_1 \in E$.*

With regard to the locally finite ϵ_0- subset of R^m, we have the following lemma which we need later.

Lemma 1.10. *Let D be a locally finite subset of an open subset E of R^m. Then D is ϵ_0-dense in E if and only if $E \subseteq \underset{b \in D}{\cup} \overline{B}(b, \epsilon_0)$.*

Proof. Since if D is closed then this lemma follows from the lemma 1.7, we need only to prove that any locally finite subset of R^m is closed. Assume that there is a locally finite subset D of R^m such that D is not closed. Then $D' - D \neq \emptyset$. Hence there exists $\alpha \in R^m$ such that $\alpha \in D' - D$. Since $\alpha \in D'$, the set $B(\alpha, \epsilon) \cap (D - \{\alpha\})$ is an infinite subset of R^m for all $\epsilon > 0$. This is a contradiction since D is locally finite. Hence we have $\overline{D} = D$. □

2. Minimization of the dense subset

In this section, we investigate the concept of the minimal ϵ_0-dense subset in R^m and research the shape of this set. Throughout this section, $\epsilon_0 > 0$ denotes any, but fixed, positive real number. Recall that a point $a \in D$ is a point of the ϵ_0-dense ace of D in E if and only if $D - \{a\}$ is not ϵ_0-dense in E.

Note that if D is an ϵ_0-dense subset of E in E for a non-empty open subset E of R^m with $\epsilon_0 > 0$ then an element $a \in D$ is a point of the ϵ_0-dense ace of D in E if and only if there is a positive real number $\epsilon_1 > \epsilon_0$ and a point $b \in E$ such that $B(b, \epsilon_1) \cap D = \{a\}$ by the theorem 3.3 in [1]. In this case, the point $b \in E$ must satisfy the relation $\|a - b\| \leq \epsilon_0$.

Definition 2.1. Let E be an open subset of R^m and D be a non-empty ϵ_0-dense subset of E. Let us denote the set of all the points of ϵ_0-dense ace of D in R^m by $dap_{\epsilon_0}(D)$ or $dap_{\epsilon_0}(D; R^m)$ and in E by $dap_{\epsilon_0}(D; E)$.

Note that $dap_{\epsilon_0}(D; E)$ is countable for any ϵ_0-dense subset D of E by the corollary 3.4 in [1].

Definition 2.2. Let E be a non-empty open subset of R^m and D be an ϵ_0-dense subset of E in E. We define that D is a minimal ϵ_0-dense subset of E in E if and only if $dap_{\epsilon_0}(D; E) = D$. And we define that an ϵ_0-dense subset D can be minimized if and only if there is a subset D_0 of D such that D_0 is a minimal ϵ_0-dense subset of D in E.

Theorem 2.3. *Let E be a non-empty open subset of R^m and D be an ϵ_0-dense subset of E in E with $\epsilon_0 > 0$. Suppose that $D - dap_{\epsilon_0}(D; E)$ is finite. Then D can be minimized.*

Proof. Let D be the given ϵ_0-dense subset of E in E. If $D = dap_{\epsilon_0}(D; E)$ then D is a minimal ϵ_0-dense subset of E in E which completes the proof since $D \subseteq D$. Now suppose that $D \neq dap_{\epsilon_0}(D; E)$. Then $D - dap_{\epsilon_0}(D; E) \neq \emptyset$. Since this set is finite, we may set $D - dap_{\epsilon_0}(D; E) = \{a_1, a_2, \cdots, a_n\}$ for some elements $a_k's$ and some natural number n. Since a_1 is not an ϵ_0-dense ace of D, $D_1 = D - \{a_1\}$ is ϵ_0-dense in E. If $D_1 - \{a_2\} = D - \{a_1, a_2\}$ is not ϵ_0-dense in E then a_2 is an ϵ_0-dense ace of the ϵ_0-dense subset $D_1 = D - \{a_1\}$. In this case, we take $D_2 = D_1$. On the other hand, if $D_1 - \{a_2\} = D - \{a_1, a_2\}$ is ϵ_0-dense in E then we take $D_2 = D_1 - \{a_2\}$. Then we have

$$D_2 - dap_{\epsilon_0}(D_2; E) \subseteq \{a_3, \cdots, a_n\}.$$

Similarly, if $D_2 - \{a_3\}$ is not ϵ_0-dense in E then a_3 is an ϵ_0-dense ace of the ϵ_0-dense subset D_2. In this case, we take $D_3 = D_2$. On the other hand, if $D_2 - \{a_3\}$ is ϵ_0-dense in E then we take $D_3 = D_2 - \{a_3\}$. Then we have $D_3 - dap_{\epsilon_0}(D_3; E) \subseteq \{a_4, \cdots, a_n\}$. Inductively, if $D_{n-1} - \{a_n\}$ is not ϵ_0-dense in E then a_n is an ϵ_0-dense ace of the ϵ_0-dense subset D_{n-1}. In this case, we take $D_n = D_{n-1}$. On the other hand, if $D_{n-1} - \{a_n\}$ is ϵ_0-dense in E then we take $D_n = D_{n-1} - \{a_n\}$. Then we have $D_n - dap_{\epsilon_0}(D_n; E) = \emptyset$. This implies that D_n is a minimal ϵ_0-dense subset of D in E which completes the proof. □

Definition 2.4. Let E be a non-empty open subset of R^m and D be a non-empty subset of E. We define that a point $a \in R^m$ is an ϵ_0-uncatchable or ϵ_0-untouchable point with respect to the subset D in E if and only if the point a is an element of the set $E' - E$ such that $a \notin \overline{B}(b, \epsilon_0)$ for all points $b \in D$.

Lemma 2.5. *Let D be a subset of an open subset E of R^m and $\epsilon_0 \geq 0$ be any, but fixed, non-negative real number. Then D is ϵ_0-dense in E if and only if $E \subseteq \bigcup_{b\in D} B(b, \epsilon)$ for each positive real number $\epsilon > \epsilon_0$.*

Proof. From lemma 1.8, D is ϵ_0-dense in E if and only if $E \subseteq \bigcup_{b\in D} \overline{B}(b, \epsilon)$ for each positive real number $\epsilon > \epsilon_0$. Hence we need only to show that $E \subseteq \bigcup_{b\in D} B(b, \epsilon)$ for each positive real number $\epsilon > \epsilon_0$ if and only if $E \subseteq \bigcup_{b\in D} \overline{B}(b, \epsilon)$ for each positive real number $\epsilon > \epsilon_0$. The sufficient condition is obvious. In order to prove the necessary condition, suppose that $E \subseteq \bigcup_{b\in D} \overline{B}(b, \epsilon)$ for each positive real number $\epsilon > \epsilon_0$. And let any positive real number $\epsilon > \epsilon_0$ be given. Since $\frac{\epsilon+\epsilon_0}{2} > \epsilon_0$, we have $E \subseteq \bigcup_{b\in D} \overline{B}(b, \frac{\epsilon+\epsilon_0}{2})$. Since $\frac{\epsilon+\epsilon_0}{2} < \epsilon$, we have

$$E \subseteq \bigcup_{b\in D} \overline{B}(b, \frac{\epsilon + \epsilon_0}{2}) \subseteq \bigcup_{b\in D} B(b, \epsilon).$$

This completes the proof. □

Lemma 2.6. *Let D be a subset of an open subset E of R^m and $\epsilon_0 \geq 0$ be any, but fixed, non-negative real number. Then D is ϵ_0-dense in E if and only if $E \subseteq \bigcup_{b\in D} B(b, \epsilon_0 + \frac{1}{p})$ for each natural number $p \in N$.*

Proof. From lemma 2.5, D is ϵ_0-dense in E if and only if $E \subseteq \bigcup_{b\in D} B(b, \epsilon)$ for each positive real number $\epsilon > \epsilon_0$. Hence we need only to show that $E \subseteq \bigcup_{b\in D} B(b, \epsilon)$ for each positive real number $\epsilon > \epsilon_0$ if and only if $E \subseteq \underset{b\in D}{\cup} B(b, \epsilon_0 + \frac{1}{p})$ for each natural number $p \in N$. The sufficient condition is obvious. In order to prove the necessary condition, suppose that any $\epsilon > \epsilon_0$ be given. Then there is a natural number $p \in N$ such that $\epsilon_0 + \frac{1}{p} < \epsilon$. Hence we have

$$E \subseteq \underset{b\in D}{\cup} B(b, \epsilon_0 + \frac{1}{p}) \subseteq \underset{b\in D}{\cup} B(b, \epsilon)$$

which completes the proof. □

LEMMA 2.7. *Let E be a non-empty open subset of R^m and D be an ϵ_0-dense subset of E in E with $\epsilon_0 > 0$. Then there is a countable subset D_0 of D such that D_0 is an ϵ_0-dense subset of D in E.*

Proof. Since E is an open subset of R^m and any closed and bounded subset of R^m is compact, there is an increasing sequence K_n of compact subsets of E such that $E = \underset{n\in N}{\cup} K_n$. By lemma 2.6, we have $E \subseteq \underset{b\in D}{\cup} B(b, \epsilon_0 + \frac{1}{p})$ for each natural number $p \in N$. Since $K_n \subseteq E$ for all natural number n, the collection $\{B(b, \epsilon_0 + \frac{1}{p}) : b \in D\}$ is an open cover of the set K_n for each natural number n. Since K_n is compact, there is a finite subcover of this collection for each natural number n. Since this holds for any natural number p and the set N is countable, for each natural number n, there is a countable subset $D_n \subseteq D$ such that $K_n \subseteq \underset{b\in D_n}{\cup} B(b, \epsilon_0 + \frac{1}{p})$ for each natural number $p \in N$. Take $D_0 = \underset{n\in N}{\cup} D_n$. Then D_0 is countable and $E \subseteq \underset{b\in D_0}{\cup} B(b, \epsilon_0 + \frac{1}{p})$ for each natural number $p \in N$. Hence D_0 is a countable ϵ_0-dense subset of D in E by the lemma just above. □

DEFINITION 2.8. Let E be a non-empty open subset of R^m and $D = \{d_n | n \in N\}$ be a countable ϵ_0-dense subset of E. If $D = \{d_1, \cdots, d_K\}$ is finite then we define that $d_n = d_K$ for all natural numbers $n \geq K$.

(a) If d_1 is an ϵ_0-dense ace of D in E then we let $D_1 = D$ and let $D_1 = D - \{d_1\}$ if d_1 is not an ϵ_0-dense ace of D. If d_2 is an ϵ_0-dense ace of D_1 in E then we take $D_2 = D_1$ and take $D_2 = D_1 - \{d_2\}$ if d_2 is not an ϵ_0-dense ace of D_1 in E. Inductively, if d_n is an ϵ_0-dense ace of D_{n-1} in E then we take $D_n = D_{n-1}$ and take $D_n = D_{n-1} - \{d_n\}$ if d_n is not an ϵ_0-dense ace of D_{n-1} in E.

(b) The subset $D_\infty = \underset{n\in N}{\cap} D_n$ is called the minimized cluster with respect to the sequence $\{d_n | n \in N\}$.

(c) The process obtaining the minimized cluster as the above is said to be the minimizing process of $\{d_n | n \in N\}$.

DEFINITION 2.9. Let $D = \{d_n | n \in N\}$ be an infinite sequence in R^m. Let $\{n_1, \cdots, n_k, \cdots\}$ be any sequence of natural numbers such that $n_1 < \cdots < n_k < \cdots$. Then a sequence $\{d_{n_k} | k \in N\}$ is said to be a head subsequence of $D = \{d_n | n \in N\}$ if and only if $\{d_{n_k} | k \in N\}$ is a subsequence of $D = \{d_n | n \in N\}$ or $d_{n_k} = d_{n_K}$ for all natural numbers $k \geq K$ for some natural number K.

THEOREM 2.10. *Let E be a non-empty open subset of R^m and $D = \{d_n | n \in N\}$ be an ϵ_0-dense subset of E in E with $\epsilon_0 > 0$. If $D = \{d_1, \cdots, d_K\}$ is finite then we define that $d_n = d_K$ for all natural numbers $n \geq K$. Then D can be minimized if and only if there is a head subsequence $\{d_{n_k}\}$ of $\{d_n | n \in N\}$ such that the minimized cluster with respect to the sequence $\{d_{n_k} | k \in N\}$ is an ϵ_0-dense subset of D in E.*

Proof. First, suppose that D can be minimized. Then there is an ϵ_0-dense subset D_z of D in E such that $D_z = dap_{\epsilon_0}(D_z; E)$. If D_z is infinite then D_z is a subsequence of D and we may set $D_z = \{d_{n_k} | k \in N\}$. If D_z is finite then there is a head subsequence $\{d_{n_k} | k \in N\}$ of D such that $D_z = \{d_{n_1}, \cdots, d_{n_K}\}$ for some natural number K. Then the minimizing cluster with respect to this head subsequence $D_z = \{d_{n_k} | k \in N\}$ is the subset D_z itself since each d_{n_k} is an ϵ_0-dense ace of $D_{n_{k-1}}$. Since D_z is ϵ_0-dense in E, this completes the proof of the sufficient condition. For the converse, suppose that there is a head subsequence $\{d_{n_k}\}$ of $\{d_n | n \in N\}$ such that the minimized cluster D_∞ with respect to the sequence $\{d_{n_k} | k \in N\}$ is an ϵ_0-dense subset of D in E. We need only to show that $D_\infty = dap_{\epsilon_0}(D_\infty; E)$. Note that if A, B are ϵ_0-dense subsets of E in E such that $A \subseteq B$ then $dap_{\epsilon_0}(B; E) \subseteq dap_{\epsilon_0}(A; E)$. Now suppose that $D_\infty \neq dap_{\epsilon_0}(D_\infty; E)$. Then there is an element $d_{n_{k_0}} \in D_\infty$ such that $d_{n_{k_0}}$ is not an ϵ_0-dense ace of D_∞ in E. Since $d_{n_{k_0}} \in D_{n_{k_0+1}}$, we have $d_{n_{k_0}} \in dap_{\epsilon_0}(D_{n_{k_0}}; E)$ by the definition of the set $D_{n_{k_0+1}}$. Hence we must have $d_{n_{k_0}} \in dap_{\epsilon_0}(D_\infty; E)$ since $D_\infty \subseteq D_{n_{k_0}}$. This contradiction completes the proof. □

THEOREM 2.11. *Let E be a non-empty open subset of R^m and $D = \{d_n | n \in N\}$ be an ϵ_0-dense subset of E in E with $\epsilon_0 > 0$. Suppose that $D - dap_{\epsilon_0}(D; E) = \{d_{n_k} | k \in N\}$. Let $C_1 = D - \{d_{n_1}\}$. If d_{n_2} is an ϵ_0-dense ace of C_1 then we take $C_2 = C_1$. On the other hand, we take $C_2 = C_1 - \{d_{n_2}\}$ if d_{n_2} is not an ϵ_0-dense ace of C_1. Inductively, if d_{n_k} is an ϵ_0-dense ace of C_{k-1} then we set $C_k = C_{k-1}$. On the other hand, we take $C_k = C_{k-1} - \{d_{n_2}\}$ if d_{n_k} is not an ϵ_0-dense ace of C_{k-1}. Then $C_0 = \bigcap_{n \in N} C_n$ is the minimized cluster with respect to the sequence*

$\{d_n | n \in N\}$ *and* $C_k = \bigcap_{n=1}^{k} C_n$ *is the minimized cluster with respect to the sequence* $\{d_n | n \in N\}$ *if* $D - dap_{\epsilon_0}(D; E) = \{d_{n_1}, \cdots, d_{n_k}\}$ *is finite.*

Proof. If $D - dap_{\epsilon_0}(D; E) = \emptyset$ then we have $D = dap_{\epsilon_0}(D; E)$ and the minimizing cluster with respect to the sequence $D = \{d_n | n \in N\}$ is the set D itself. Now suppose that $D - dap_{\epsilon_0}(D; E) \neq \emptyset$. Then $D - dap_{\epsilon_0}(D; E)$ is finite or infinite. Suppose that $D - dap_{\epsilon_0}(D; E) = \{d_{n_1}, \cdots, d_{n_k}, \cdots\}$ is infinite. Since all the elements of $\{d_1, \cdots, d_{n_1-1}\}$ are the ϵ_0-dense aces of D, we have $D = D_1 = D_2 = \cdots = D_{n_1-1}$ in the minimizing process of $\{d_n | n \in N\}$. Since d_{n_1} is not an ϵ_0-dense ace of $D = D_{n_1-1}$, we have $D_{n_1} = D_{n_1-1} - \{d_{n_1}\} = D - \{d_{n_1}\} = C_1$. Similarly, since all the elements of $\{d_{n_1+1}, \cdots, d_{n_2-1}\}$ are the ϵ_0-dense aces of D and $D - \{d_{n_1}\} = C_1$, we have $D_{n_1} = D_{n_1+1} = \cdots = D_{n_2-1} = C_1$ in the minimizing process. Now if d_{n_2} is an ϵ_0-dense ace of $D_{n_2-1} = C_1$, then we have $D_{n_2} = D_{n_2-1} = C_1 = C_2$. On the other hand, if d_{n_2} is not an ϵ_0-dense ace of $D_{n_2-1} = C_1$, then we have $D_{n_2} = D_{n_2-1} - \{d_{n_2}\} = C_1 - \{d_{n_2}\} = C_2$. Inductively, since all the elements of $\{d_{n_{k-1}+1}, \cdots, d_{n_k-1}\}$ are the ϵ_0-dense aces of D and $D_{n_{k-1}} = C_{k-1}$, we have $D_{n_{k-1}} = D_{n_{k-1}+1} = \cdots = D_{n_k-1} = C_{k-1}$. If d_{n_k} is an ϵ_0-dense ace of $D_{n_k-1} = C_{k-1}$, then we have $D_{n_k} = D_{n_k-1} = C_{k-1} = C_k$. On the other hand, if d_{n_k} is not an ϵ_0-dense ace of $D_{n_k-1} = C_{k-1}$, then we have $D_{n_k} = D_{n_k-1} - \{d_{n_k}\} = C_{k-1} - \{d_{n_k}\} = C_k$. Therefore, we have $D_\infty = \bigcap_{n \in N} D_n = \bigcap_{k \in N} D_{n_k} = \bigcap_{k \in N} C_k = C_0$. This is also true in the case where $D - dap_{\epsilon_0}(D; E) = \{d_{n_1}, \cdots, d_{n_k}\}$ is finite since all the elements of $\{d_{n_k+1}, d_{n_k+2}, d_{n_k+3}, \cdots\}$ are the ϵ_0-dense aces of $D_{n_p} = C_k$ for all natural number $p \geq k$. This completes the proof. □

Example 2.12. *Let* E *be the open subset of* R^2 *such that* $E = \bigcup_{n \in N} B((-\frac{1}{2^{n-1}}, 0), 1)$ *and* $D = \{(-\frac{1}{2^{n-1}}, 0) : n \in N\}$*. Then* $\overline{D} = D \cup \{(0, 0)\}$*. Hence we have* $E \subseteq \bigcup_{b \in \overline{D}} \overline{B}(b, 1)$*. Thus* D *is an* $1-$*dense subset of* E *by the lemma 1.7. Now we claim that* $(-\frac{1}{2^{n-1}}, 0)$ *is an* $1-$*dense ace of* D *in* E *for all* $n \in N$*. Clearly,* $(-1, 0)$ *is 1-dense ace. For each natural number* $n \in N$*, consider the element* $(-\frac{1}{2^n}, 1 - \frac{1}{p})$ *with natural number* $p \in N$*. We have* $(-\frac{1}{2^n}, 1 - \frac{1}{p}) \in E$ *for all natural number* $p \in N$*. Choose a natural number* p_0 *so large that* $\sqrt{(\frac{1}{2^{n+1}})^2 + (1 - \frac{1}{p_0})^2} > 1$*. The distance between* $(-\frac{1}{2^{n+1}}, 0)$ *and* $(-\frac{1}{2^n}, 1 - \frac{1}{p_0})$ *is given by* $\sqrt{(\frac{1}{2^{n+1}})^2 + (1 - \frac{1}{p_0})^2}$ *which is greater than* 1*. Hence the point* $(-\frac{1}{2^n}, 1 - \frac{1}{p_0})$ *can not be a* *point in the union* $\bigcup_{b\in\overline{D}-\{(-\frac{1}{2^n},0)\}} \overline{B}(b,1)$. *Hence* $(-\frac{1}{2^n},0)$ *is 1-dense ace in* E *for all natural number* $n \in N$. *Therefore,* D *is the minimal* 1-*dense subset of* D *in* E. *Note that the point* $(1,0)$ *is an untouchable point of* E. *Hence an* ϵ_0-*dense subset* D *can be minimized even if there is an untouchable point with respect to* D.

THEOREM 2.13. *Let* E *be a non-empty open subset of* R^m *and* D *be a locally finite* ϵ_0-*dense subset of* E *in* E *with* $\epsilon_0 > 0$. *Then* D *can be minimized.*

Proof. Let D be a locally finite ϵ_0-dense subset of E in E with $\epsilon_0 > 0$. If E is bounded then D must be a finite ϵ_0-dense subset in E. Hence D can be minimized by the theorem 2.3 since $D - dap_{\epsilon_0}(D;E)$ is finite. Hence we need only to prove the conclusion in case where E is unbounded and $D = \{d_n | n \in N\}$ with $\|d_n\| \leq \|d_{n+1}\|$. Then we have $\lim_{n\to\infty} \|d_n\| = \infty$ since E is unbounded. Now let D_∞ be the minimized cluster with respect to the sequence $\{d_n | n \in N\}$. By theorem 2.10, we need only to show that D_∞ is ϵ_0-dense in E. Let any element $x_0 \in E$ be given. Then there is a natural number $K \in N$ with $K > 1$ such that $\|x_0\| + 4\epsilon_0 \leq \|d_K\|$ since $\lim_{n\to\infty} \|d_n\| = \infty$. Consider the set D_K in the minimizing process of D which was introduced in the definition 2.8. The set D_K is ϵ_0-dense subset of D in E and is closed since D is locally finite. Hence there is an element d_{n_0} of D_K such that $x_0 \in \overline{B}(d_{n_0}, \epsilon_0)$. Then we have $\|d_{n_0}\| \leq \|d_{n_0} - x_0\| + \|x_0\| \leq \epsilon_0 + \|x_0\| < \|d_K\|$. Thus we have $n_0 < K$. But this implies that $d_{n_0} \in D_{n_0}$. Hence d_{n_0} is an ϵ_0-dense ace of D_{n_0-1}. Thus $d_{n_0} \in D_\infty$. Therefore, $x_0 \in \bigcup_{b\in D_\infty} \overline{B}(b,\epsilon_0)$ and $E \subseteq \bigcup_{b\in D_\infty} \overline{B}(b,\epsilon_0)$. Since D_∞ is locally finite subset of R^m, this implies that the minimized cluster D_∞ is an ϵ_0-dense subset of D in E by the lemma 1.10 which completes the proof. □

THEOREM 2.14. *Let* E *be a non-empty open subset of* R^m *and* D *be a subset of* E *such that* $E \subseteq \bigcup_{b\in D} \overline{B}(b,\epsilon_0)$ *with* $\epsilon_0 > 0$. *Suppose that there is no* ϵ_0-*untouchable point with respect to* D *in* E. *Then there exists a minimal* $\epsilon-$*dense subset of* D *in* E *for each positive real number* $\epsilon > \epsilon_0$.

Proof. We first claim that $E' \subseteq \bigcup_{b\in D} \overline{B}(b,\epsilon_0)$. Suppose that this is not true. Then there exists a point $x_0 \in E'$ such that $x_0 \notin \bigcup_{b\in D} \overline{B}(b,\epsilon_0)$. Since $E \subseteq \bigcup_{b\in D} \overline{B}(b,\epsilon_0)$, we have $x_0 \in E' - E$ and $x_0 \notin \overline{B}(b,\epsilon_0)$ for each element $b \in D$. Hence x_0 is an ϵ_0-untouchable point with respect to D

in E. This contradiction implies that $\overline{E} \subseteq \underset{b\in D}{\cup} \overline{B}(b, \epsilon_0)$. Let any positive real number $\epsilon > \epsilon_0$ be given. Now we have

$$\overline{E} \cap \overline{B}(0, 4^k\epsilon_0) \subseteq \underset{b\in D}{\cup} \overline{B}(b, \epsilon_0) \subseteq \underset{b\in D}{\cup} B(b, \epsilon)$$

for each natural number $k \in N$. Since the subset $\overline{E} \cap \overline{B}(0, 4\epsilon_0)$ is compact, there are some finite points $d_1, \cdots, d_{n_1} \in D$ such that $\overline{E} \cap \overline{B}(0, 4\epsilon_0) \subseteq \overset{n_1}{\underset{k=1}{\cup}} B(d_k, \epsilon)$. Inductively, since the subset $\overline{E} \cap \overline{B}(0, 4^k\epsilon_0)$ is compact, there are some finite points $d_{n_{k-1}+1}, \cdots, d_{n_k} \in D$ such that $\overline{E} \cap \overline{B}(0, 4^k\epsilon_0) \subseteq \overset{n_k}{\underset{i=1}{\cup}} B(d_i, \epsilon)$. Hence we have $\overline{E} \subseteq \overset{\infty}{\underset{k=1}{\cup}} B(d_k, \epsilon)$. Thus we have $E \subseteq \overline{E} \subseteq \overset{\infty}{\underset{k=1}{\cup}} \overline{B}(d_k, \epsilon)$. Since $\{d_k\}_{k=1}^{\infty}$ is locally finite, this implies that $\{d_k\}_{k=1}^{\infty}$ is a locally finite $\epsilon-$dense subset of D in E by the lemma 1.6. Therefore, $\{d_k\}_{k=1}^{\infty}$ can be minimized by the theorem just above. □

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